AP® CALCULUS AB/CALCULUS BC
2014 SCORING GUIDELINES

Question 1

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

(a) $\dfrac{A(30) - A(0)}{30 - 0} = -0.197$ (or -0.196) lbs/day	1 : answer with units
(b) $A'(15) = -0.164$ (or -0.163) The amount of grass clippings in the bin is decreasing at a rate of 0.164 (or 0.163) lbs/day at time $t = 15$ days.	$2 : \begin{cases} 1 : A'(15) \\ 1 : \text{interpretation} \end{cases}$
(c) $A(t) = \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \Rightarrow t = 12.415$ (or 12.414)	$2 : \begin{cases} 1 : \dfrac{1}{30}\displaystyle\int_0^{30} A(t)\,dt \\ 1 : \text{answer} \end{cases}$
(d) $L(t) = A(30) + A'(30) \cdot (t - 30)$ $A'(30) = -0.055976$ $A(30) = 0.782928$ $L(t) = 0.5 \Rightarrow t = 35.054$	$4 : \begin{cases} 2 : \text{expression for } L(t) \\ 1 : L(t) = 0.5 \\ 1 : \text{answer} \end{cases}$

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

   (a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

$$\frac{A(30)-A(0)}{30-0} = \frac{-5.904}{30} \approx -.197 \ \frac{\text{pounds}}{\text{day}}$$

   (b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

$$A'(t) \approx -.478(.931)^t$$

$$A'(15) \approx -.164 \ \frac{\text{pounds}}{\text{day}}$$

The amount of grass clippings in the bin is decreasing (decomposing) at a rate of .164 pounds per day at time = 15 days

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Continue problem 1 on page 5.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

average amount $= \frac{1}{30}\int_0^{30} A(t)\,dt \approx 2.75263511$

$A(t) = 2.7526351 = 6.687(.931)^t$

this occurs at $t \approx 12.414$ days

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

$L(t)$ is the tangent line to $A(t)$ at $t=30$

$A(30) \approx .783 \Rightarrow (30, .783) = (t, A(t))$

$A'(30) \approx -.056$ let $-.056 = M$

$(y - y_1) = M(x - x_1)$ so for this problem,

$(A(t) - .783) = -.056(t - 30)$

when there are .5 pounds of grass, $A(t) = .5$,

$(.5 - .783) = -.056(t - 30)$

$t = 35.054$ days

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GO ON TO THE NEXT PAGE.

 1B1

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

$$\frac{A(30)-A(0)}{30-0} = \frac{.7829278-6.687}{30} = -.196\ \frac{\text{pounds}}{\text{day}}$$

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

$$A'(t) = -.47809(.931)^x \big|_{x=15} = -.163\ \frac{\text{pounds}^2}{\text{day}}$$

this value represents the rate of which the grass is decomposing in pound² per day @ $t=15$ days

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Continue problem 1 on page 5.

1B₂

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

$\frac{1}{30-0}\int_0^{30} A(t)\,dt = 2.752$

$2.752 = 6.687(.931)^t$

$t = 12.418$

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(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

$A(30) = .783$

$(30, .783)$

$A'(30) = -.47809(.931)^{30} = -.056$

$L(t) - .783 = -.056(t-30)$

$L(t) = -.056t + 1.68 + .783$

$L(t) = -.056t + 2.463$

$.5 = -.056t + 2.463$

$t = 35.053$

GO ON TO THE NEXT PAGE.

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

in bin

$a^u = \ln a \cdot a^u \cdot u'$

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

$\frac{1}{30-0}\int_0^{30} A'(t)\,dt = \frac{1}{30} \cdot 6.687 \cdot \int_0^{30} [(\ln .931)(.931^t)]\,dt$

$= -.1968024044$ lbs of grass/day

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

$A'(t) = 6.687 \cdot \ln(.931) \cdot (.931^t)$

$A'(15) = 6.687 \cdot \ln(.931) \cdot (.931^{15}) = -.1635905804$

the rate at which the amount of grass in the bin is decomposing is $-.1635905805$ pounds per day

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Continue problem 1 on page 5.

SSS$_2$

IC$_2$

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

$$A(t) = \frac{1}{30}\int_0^{30} A(t)\,dt$$

$$6.687(.931)^t = \frac{1}{30}\left[(6.687(.931)^{30}) - (6.687(.931)^{0})\right]$$

$$6.687(.931)^t = 24981.18241$$

~~there~~ is not t

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(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

$$.5 = L(t)$$

GO ON TO THE NEXT PAGE.

Question 1

Overview

In this problem students were given $A(t)$, a model for the amount of grass clippings, in pounds, contained in a bin at time t days for $0 \le t \le 30$. In part (a) students were asked to show the calculation of the average rate of change of $A(t)$ over the interval $0 \le t \le 30$ and specify the units of the result – pounds per day. In part (b) students were asked to calculate the derivative of $A(t)$ at $t = 15$, either by using the calculator or by applying basic derivative formulas to $A(t)$ to obtain $A'(t)$ and then evaluating $A'(t)$ at $t = 15$. This answer is negative. Therefore, students needed to interpret the absolute value of this answer as the rate at which the amount of grass clippings in the bin is decreasing, in pounds per day, at time $t = 15$ days. In part (c) students were given two tasks. First, students needed to set up and evaluate the integral expression for the average value of $A(t)$ over the interval $0 \le t \le 30$, namely $\frac{1}{30}\int_0^{30} A(t)\,dt$. Second, students needed to set up and solve the equation $A(t) = \frac{1}{30}\int_0^{30} A(t)\,dt$ for t in the interval $0 \le t \le 30$. In part (d) students needed to compute $A(30)$, $A'(30)$, and write $L(t) = A(30) + A'(30)(t - 30)$. Students were to then solve the equation $L(t) = 0.5$.

Sample: 1A
Score: 9

The student earned all 9 points. In part (d) the student evidently stored more accurate intermediate values in the calculator because the correct answer is presented.

Sample: 1B
Score: 6

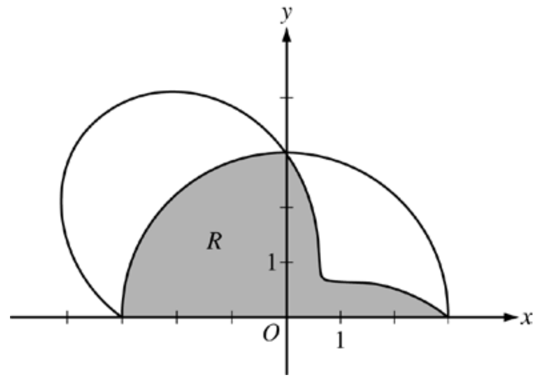
The student earned 6 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point for $A'(15)$. The units are incorrect, so the interpretation point was not earned. The student's description seems to suggest that the student thinks that $A'(15)$ is a rate of a rate or that $A'(15)$ suggests that A is decreasing at a negative rate at $t = 15$. In part (c) the student earned the first point for $\frac{1}{30-0}\int_0^{30} A(t)\,dt$. The student's answer is not accurate to three decimal places. In part (d) the student earned the first two points for $L(t)$ and the point for setting $L(t) = 0.5$. The student's answer is not accurate to three decimal places.

Question 1 (continued)

Sample: 1C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point for $A'(15)$. The interpretation point was not earned because the student's claim that the grass clippings decompose at a negative rate is incorrect. The grass clippings are decreasing at the rate of $|A'(15)|$ pounds per day at $t = 15$. In part (c) the student earned the first point for $\frac{1}{30}\int_0^{30} A(t)\, dt$. In part (d) the student is not eligible for the point for setting $L(t) = 0.5$ because neither of the first 2 points was earned.

Question 2



The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$.

Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

(a) $$\text{Area} = \frac{9\pi}{4} + \frac{1}{2}\int_0^{\pi/2} (3 - 2\sin(2\theta))^2 \, d\theta$$
$$= 9.708 \text{ (or } 9.707\text{)}$$

$$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$$

(b) $$x = (3 - 2\sin(2\theta))\cos\theta$$
$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = -2.366$$

$$2 : \begin{cases} 1 : \text{expression for } x \\ 1 : \text{answer} \end{cases}$$

(c) The distance between the two curves is
$D = 3 - (3 - 2\sin(2\theta)) = 2\sin(2\theta)$.

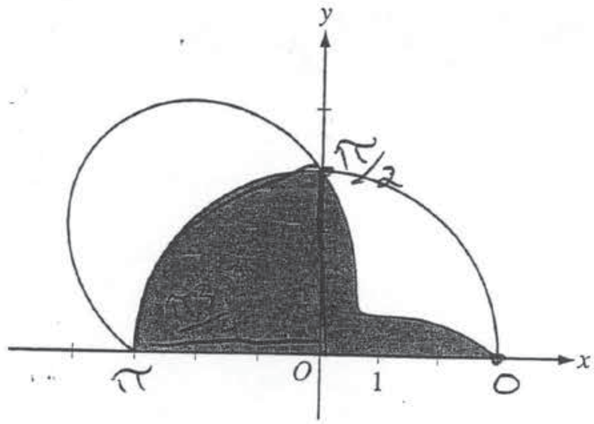
$$\left.\frac{dD}{d\theta}\right|_{\theta=\pi/3} = -2$$

$$2 : \begin{cases} 1 : \text{expression for distance} \\ 1 : \text{answer} \end{cases}$$

(d) $$\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dr}{d\theta} \cdot 3$$
$$\left.\frac{dr}{dt}\right|_{\theta=\pi/6} = (-2)(3) = -6$$

$$2 : \begin{cases} 1 : \text{chain rule with respect to } t \\ 1 : \text{answer} \end{cases}$$

2A₁



2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

   (a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

$\cancel{\frac{9\pi}{4}}$

$$3 = 3 - 2\sin(2\theta)$$
$$0 = 2\sin(2\theta)$$
$$0 = \sin(2\theta)$$
$$2\theta = 0, \pi, 2\pi$$
$$\theta = 0, \frac{\pi}{2}, \pi$$

$$\frac{9\pi}{4} + \frac{1}{2}\int_0^{\pi/2} (3 - 2\sin(2\theta))^2 \, d\theta = \boxed{9.707963268}$$

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Continue problem 2 on page 7

2A₂

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

$x = r\cos\theta$

$x = (3 - 2\sin(2\theta))\cos\theta$

$\frac{dx}{d\theta}\Big|_{\theta=\frac{\pi}{6}} \left[(3-2\sin(2\theta))\cos\theta\right] = \boxed{-2.366025009}$

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

$3 - (3 - 2\sin(2\theta))$

$\frac{d}{d\theta} 2\sin(2\theta) = 2 \cdot 2\cos(2\theta)$

$4\cos(2\theta)$

$4\cos(2\frac{\pi}{3}) = \boxed{-2}$

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \geq 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

$\frac{dr}{dt} = -2\cos(2\theta) \cdot 2\frac{d\theta}{dt}$

$\frac{dr}{dt} = -2\cos\left(\frac{2\pi}{6}\right) \cdot 2(3)$

$\frac{dr}{dt} = -6$

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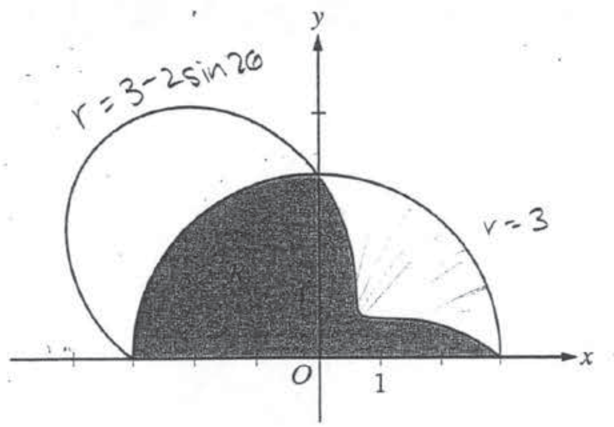
2B.



2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

$\frac{1}{4}(\pi r^2) = \frac{1}{4}\pi(3)^2 = \frac{9}{4}\pi$

$A = \frac{9}{4}\pi + \pi\int_0^{\pi/2}((3)^2 - (3-2\sin 2\theta)^2)\,d\theta$

$\boxed{A = 34.898}$

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Continue problem 2 on page 7.

2B₂

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

$x = r\cos\theta$

$$\frac{dx}{d\theta} = -r\sin\theta + r'\cos\theta$$

$$\frac{dx}{d\theta} = -(3 - 2\sin 2\theta)\sin\theta + -4\cos 2\theta \cdot \cos\theta$$

$$\frac{dx}{d\theta} \text{ at } \pi/6 = \boxed{-2.366}$$

$-.63397$

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

$y = r\sin\theta$

$$dy/dx = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$$

-1.5

$$dy/dx = \frac{-2\sin 2\theta\cos\theta - 4\cos 2\theta\sin\theta}{2\sin 2\theta\sin\theta - 4\cos 2\theta\cos\theta} \quad ; \theta = \pi/3$$

$$\boxed{dy/dx = 2.8868}$$

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \geq 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

$$\frac{dr}{d\theta} = -4\cos 2\theta \qquad \frac{dr}{dt} = \frac{dr/d\theta}{dt/d\theta} = \frac{-4\cos 2\theta}{1/3}$$

$$\frac{d\theta}{dt} = t$$

$$(-4\cos(\pi/3))\,3 = \boxed{-6}$$

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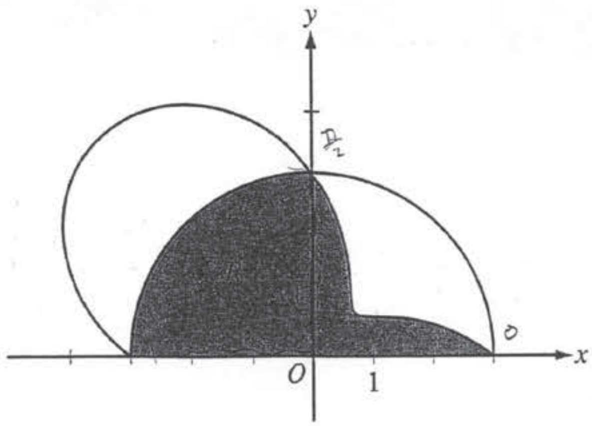
2C1



2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

   (a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

$\frac{\pi (3)^2}{4}$

$\frac{9\pi}{4} + \int_0^{\frac{\pi}{2}} 3 - 2\sin(2\theta)$

$\boxed{\approx 9.781}$

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Continue problem 2 on page 7.

2C$_2$

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

$x = r\cos\theta$

$x = (3 - 2\sin(2\theta))\cos\theta$

$\frac{Dx}{D\theta} = -4\cos(2\theta)(-\sin\theta)$

$4\cos(2\theta)\sin(\theta)$

$4\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$

$4\left(\frac{1}{2}\right)\frac{1}{2} = \boxed{1}$

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

$x = r\cos\theta$ $\quad y = r\sin\theta$

$x = 3 - 2\sin(2\theta)\cos\theta$ $\quad y = 3 - 2\sin(2\theta)\sin\theta$

$x' = \approx 2.500$ $\quad y' \approx .866$

$\sqrt{(x')^2 + (y')^2}$

$\sqrt{6.250 + 0.750}$

$\boxed{\approx 2.646}$ units

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \geq 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.

?

$\frac{DR}{Dt} = -4\cos(2\theta)\frac{D\theta}{Dt}$

$\frac{DR}{Dt} = -4\cos\left(\frac{\pi}{3}\right)3$

$-4\left(\frac{1}{2}\right)3$

$\boxed{\frac{DR}{Dt} = -6}$

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Question 2

Overview

In this problem students were given the graphs of the polar curves $r = 3 - 2\sin(2\theta)$ and $r = 3$ for $0 \le \theta \le \pi$. In part (a) students had to find the area of the shaded region R that is the common area inside both graphs. Students needed to find the area bounded by the polar curve $r = 3 - 2\sin(2\theta)$ in the first quadrant and add it to the area of the quarter circle in the second quadrant resulting in $\frac{1}{2}\int_0^{\pi/2} (3 - 2\sin(2\theta))^2 \, d\theta + \frac{9\pi}{4}$. In part (b) students needed to find $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$. Students had to realize that $x = r\cos(\theta)$ and then differentiate to find $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$. In part (c) students were asked to find the rate at which the distance between the curves in the first quadrant was changing at $\theta = \frac{\pi}{3}$. Students needed to set up an equation for the distance between the curves in the first quadrant, $D = 3 - (3 - 2\sin(2\theta))$, and then evaluate the derivative of D at the required value. Finally, in part (d) students were given that $\frac{d\theta}{dt} = 3$ and were asked to find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$. Students had to invoke the chain rule to get $\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$ and evaluate the result at $\theta = \frac{\pi}{6}$.

Sample: 2A
Score: 9

The student earned all 9 points.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student writes a correct integrand for the unshaded portion in the first quadrant. The limits of 0 and $\frac{\pi}{2}$ are correct for that integrand. The student earned the integrand and limits points. The student uses an incorrect constant on the integral and appears to be combining the integral with the shaded quarter circle in the second quadrant. The student did not earn the answer point. In part (b) the student's work is correct. In part (c) the student's work does not present a valid approach to the question. In part (d) the student's work is correct.

Sample: 2C
Score: 3

The student earned 3 points: no points in part (a), 1 point in part (b), no points in part (c), and 2 points in part (d). In part (a) the student does not square the integrand, so the integrand point was not earned. Because this is not a valid integrand for polar area, the student is not eligible for the limits and answer points. In part (b) the student earned the first point with a correct expression for x on the second line of work. The student has an incorrect value for $\frac{dx}{d\theta}$. In part (c) the student's work does not present a valid approach to the question. In part (d) the student's work is correct.

Question 3

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above.

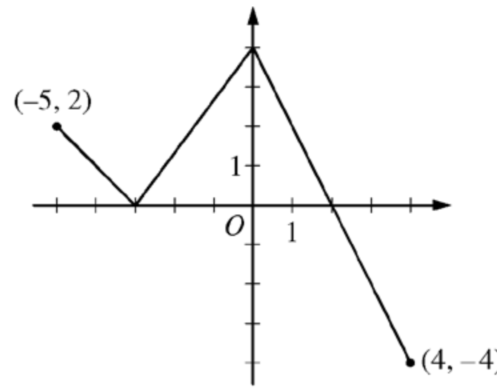
Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\, dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \dfrac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f\left(x^2 - x\right)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.



Graph of f

(a) $g(3) = \int_{-3}^{3} f(t)\, dt = 6 + 4 - 1 = 9$

1 : answer

(b) $g'(x) = f(x)$

The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

(c) $h'(x) = \dfrac{5xg'(x) - g(x)5}{(5x)^2} = \dfrac{5xg'(x) - 5g(x)}{25x^2}$

$$h'(3) = \frac{(5)(3)g'(3) - 5g(3)}{25 \cdot 3^2}$$
$$= \frac{15(-2) - 5(9)}{225} = \frac{-75}{225} = -\frac{1}{3}$$

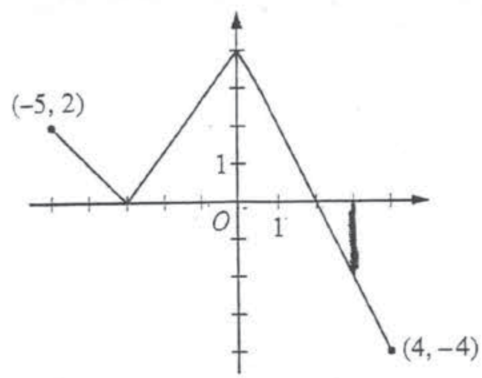
$3 : \begin{cases} 2 : h'(x) \\ 1 : \text{answer} \end{cases}$

(d) $p'(x) = f'\left(x^2 - x\right)(2x - 1)$

$p'(-1) = f'(2)(-3) = (-2)(-3) = 6$

$3 : \begin{cases} 2 : p'(x) \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

$$g(3) = \int_{-3}^{3} f(t)\,dt$$

$$= \tfrac{1}{2}(5)(4) - \tfrac{1}{2}(1)(2)$$

$$= 10 - 1 = \boxed{9}$$

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$$g'(x) > 0 \Leftrightarrow f(x) > 0$$

$$g''(x) < 0 \Leftrightarrow f'(x) < 0$$

$$(-5, -3), (0, 2)$$

Continue problem 3 on page 15.

NO CALLATOR ALLOWED

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$h(x) = \frac{g(x)}{5x}$ $\quad h'(x) = \frac{(5x)(g'(x)) - g(x)\cdot 5}{25x^2}$

$h'(3) = \frac{(15)(f(3)) - g(3)\cdot 5}{9\cdot 25} = \boxed{\frac{(15)(-2) - (9)(5)}{9\cdot 25}}$

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$p(x) = f(x^2 - x)$

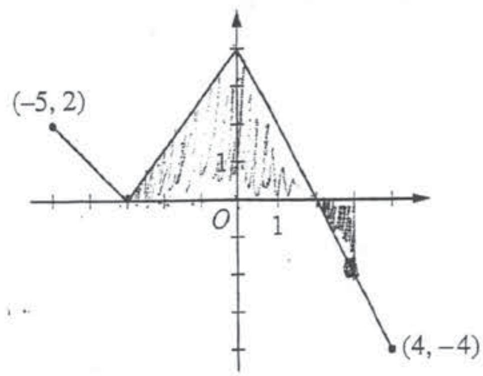
$p'(x) = f'(x^2 - x)\cdot(2x-1)$

$p'(-1) = f'(1+1)\cdot(-2-1) = f'(2)\cdot -3$

$f'(2) = \frac{-4-4}{4-0} = \frac{-8}{4} = -2$ $\quad = (-2)(-3)$

$= \boxed{6}$

NO CALCULATOR ALLOWED



Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

$\int_{-3}^{3} f(t)\,dt.$

$5 \cdot 4 = \frac{20}{2} = 10 + \frac{1(2)}{2}$

$= 11$

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$g'(x) = f(x)$

$(0, 2)$

because g' is both positive and decreasing

when g' is positive g is incr.

when g' is dec g is concave down

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Continue problem 3 on page 15.

NO CALCULATOR ALLOWED

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$g'(3) = -2$
$g(3) = 11$

$$h'(x) = \frac{5x(g'(x)) - g(x) \cdot 5}{25x^2}$$

$$h'(3) = \frac{5(3)(g'(3)) - g(3) \cdot 5}{25(9)}$$

$$h'(3) = \frac{15(-2) - 11(5)}{225}$$

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(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$p'(x) = f'(x^2 - x) \cdot (2x - 1)$

$p'(-1) = f'(0)(-3)$

$p'(-1) = -18$

$f'(0) = g(0)$
$g(0) = 6$
$f'(0) = 6$

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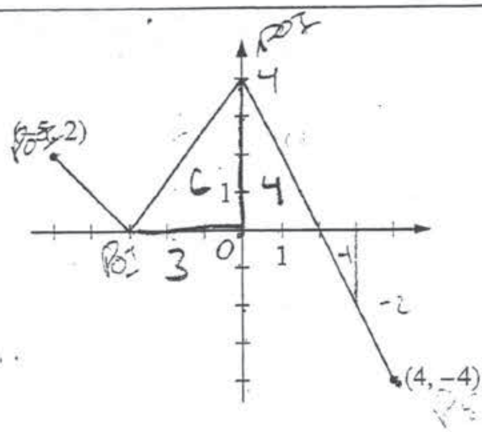

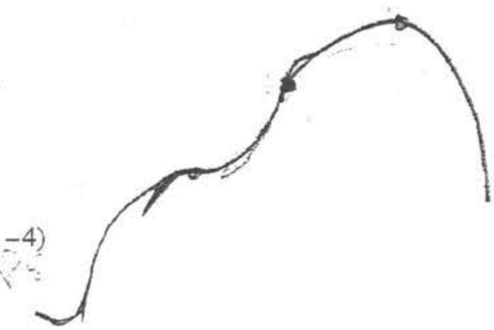

Graph of f

3. The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\, dt$.

(a) Find $g(3)$.

$g(3) = 10 + \int_2^3 -2x+4 \; dx \qquad -x^2+4x \Big|_2^3 = (-9+12)-(-4+8)$

$= 3-4 = -1$

$g(3) = 9$

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

$(-5,-2)\ (0,2)$

In these x-values f(x) is positive and the slopes of f(x) show that they are concave down because they are positive

Continue problem 3 on page 15.

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$$h'(x) = \frac{5x f(x) - 5g(x)}{25x^2}$$

$$h'(3) = \frac{15f(3) - 5(g(3))}{25 \cdot 9} = \frac{-30 - 40}{25 \cdot 9}$$

$$h'(3) = \boxed{\frac{-70}{225}}$$

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$$f(x) = \frac{4}{3}x + 4$$

$$f(x^2 - x) = \frac{4}{3}(x^2 - x) + 4$$

$$p(x) = \frac{4}{3}(x^2 - x) + 4$$

$$p'(x) = \frac{4}{3}(2x - 1)$$

$$m_{t\, x=-1} = \frac{4}{3}(2(-1) - 1) = \boxed{-4}$$

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Question 3

Overview

In this problem students were given the graph of a piecewise continuous function f defined on the closed interval $[-5, 4]$. The graph of f consists of line segments whose slopes can be determined precisely. A second function g is defined by $g(x) = \int_{-3}^{x} f(t)\, dt$. In part (a) students must calculate $g(3) = \int_{-3}^{3} f(t)\, dt$ by using a decomposition of $\int_{-3}^{3} f(t)\, dt$, such as $\int_{-3}^{3} f(t)\, dt = \int_{-3}^{2} f(t)\, dt + \int_{2}^{3} f(t)\, dt$, and by applying the relationship between the definite integral of a continuous function and the area of the region between the graph of that function and the x-axis. In part (b) students were expected to apply the Fundamental Theorem of Calculus to conclude that $g'(x) = f(x)$ on the interval $[-5, 4]$. Students were to then conclude that $g''(x) = f'(x)$ wherever $f'(x)$ is defined on $[-5, 4]$. Students needed to explain that the intervals $(-5, -3)$ and $(0, 2)$ are the only open intervals where both $g'(x) = f(x)$ is positive and decreasing. In part (c) students were expected to apply the quotient rule to find $h'(3)$ using the result from part (a) and the value $g'(3) = f(3)$ from the graph of f. In part (d) students were expected to apply the chain rule to find $p'(-1)$. This required finding $f'(2)$ from the graph of f.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: no points in part (a), 1 point in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the student reports an incorrect value for $g(3)$. In part (b) the student gives an incomplete answer, but the student is eligible for and earned the reason point. In part (c) the student's work is correct based on the imported incorrect value for $g(3)$. In part (d) the student earned both derivative points but reports an incorrect value of $p'(-1)$.

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), no points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student provides values outside of the given intervals, so the student is not eligible for the reason point. In part (c) the student's derivative is correct, but the answer is incorrect. In part (d) the student presents an incorrect expression for $p(x)$ near $x = -1$, so the student is not eligible for any points.

Question 4

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	–120	–150

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

(a) average accel $= \dfrac{v_A(8) - v_A(2)}{8 - 2} = \dfrac{-120 - 100}{6} = -\dfrac{110}{3}$ m/min^2

1 : average acceleration

(b) v_A is differentiable $\Rightarrow$ v_A is continuous

$v_A(8) = -120 < -100 < 40 = v_A(5)$

Therefore, by the Intermediate Value Theorem, there is a time t, $5 < t < 8$, such that $v_A(t) = -100$.

$2 : \begin{cases} 1 : v_A(8) < -100 < v_A(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$

(c) $s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = 300 + \int_2^{12} v_A(t)\,dt$

$$\int_2^{12} v_A(t)\,dt \approx 3 \cdot \frac{100 + 40}{2} + 3 \cdot \frac{40 - 120}{2} + 4 \cdot \frac{-120 - 150}{2} = -450$$

$s_A(12) \approx 300 - 450 = -150$

The position of Train A at time $t = 12$ minutes is approximately 150 meters west of Origin Station.

$3 : \begin{cases} 1 : \text{position expression} \\ 1 : \text{trapezoidal sum} \\ 1 : \text{position at time } t = 12 \end{cases}$

(d) Let x be train A's position, y train B's position, and z the distance between train A and train B.

$z^2 = x^2 + y^2 \Rightarrow 2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$

$x = 300$, $y = 400 \Rightarrow z = 500$

$v_B(2) = -20 + 120 + 25 = 125$

$500\dfrac{dz}{dt} = (300)(100) + (400)(125)$

$\dfrac{dz}{dt} = \dfrac{80000}{500} = 160$ meters per minute

$3 : \begin{cases} 2 : \text{implicit differentiation of distance relationship} \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters / minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\frac{v(8) - v(2)}{8-2} \rightarrow \frac{-120 - 100}{6} \rightarrow \frac{-220}{6} \rightarrow \boxed{\frac{-110}{3} \text{ m/min}^2}$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

Yes, because $v(8) = -120$ and $v(5) = 40$ and the function is differentiable and thus continuous, the train's velocity must be -100 m/min at some point between $5 < t < 8$ according to the intermediate value theorem.

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Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

$$x(12) = \int_2^{12} V_A(t)\,dt + x(2) \rightarrow \boxed{x(12) = \int_2^{12} V_A(t)\,dt + 300}$$

$$x(12) \approx 3\cdot\tfrac{1}{2}\cdot(140) + 3\cdot\tfrac{1}{2}\cdot(-80) + 4\cdot\tfrac{1}{2}\cdot(-270) + 300$$

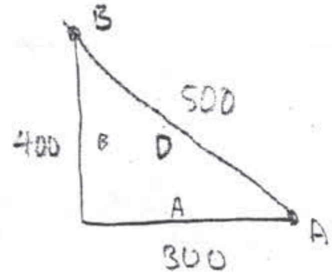
$$210 - 120 - 540 + 300$$

$$210 - 240 - 120$$

$$-30 - 120$$

$\boxed{-150}$ meaning it is 150 m W of origin station

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.



$\frac{dB}{dt} = -5t^2 + 60t + 25 \rightarrow 125$ m/min

$-20 + 120 + 25$

$120 + 5 \rightarrow 125$

$\frac{dA}{dt} = 100$ m/min

$125 \times 800 = 100000$

$$A^2 + B^2 = D^2$$

$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2D\frac{dD}{dt}$$

$$600(100) + 800(125) = 1000\frac{dD}{dt}$$

$$60000 + 100000 = 1000\frac{dD}{dt}$$

$$\frac{160{,}000}{1000} = \frac{dD}{dt} \rightarrow \boxed{\frac{dD}{dt} = 160 \text{ m/min}}$$

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NO CALCULATOR ALLOWED

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters / minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\bar{a} = \frac{V_A(8) - V_A(2)}{8-2} = \frac{-120-100}{4} = \frac{-220}{4}$$

$$= \boxed{-55 \text{ meters/minute}^2}$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

Yes, since $V_A(t)$ is continuous and differentiable, the velocity of train A must at some time t with $5 < t < 8$ equal -100 meters/minute because $V_A(5) = 40$ and $V_A(8) = -120$.

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Continue problem 4 on page 17.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

$$X_A(12) = 300 + \int_2^{12} V_A(t)\,dt$$

$$= 300 + 3(100+40) + 3(40+(-120)) + 4(-120-150)$$

$$= 300 + 420 - 240 - 1080$$

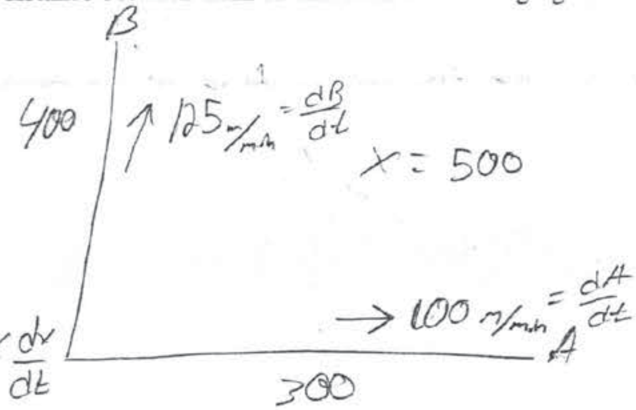
$$= \boxed{-600 \text{ meters west of the Origin Station}}$$

$$\begin{array}{r} 270 \\ \times\ 4 \\ \hline 1080 \end{array} \qquad \begin{array}{r} 1080 \\ +240 \\ \hline -1320 \\ +720 \\ \hline -600 \end{array}$$

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

$$V_B(2) = -5(2)^2 + 60(2) + 25$$

$$= 125 \text{ m/min}$$



$$A^2 + B^2 = \cancel{C}\,x^2$$

$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2x\frac{dx}{dt}$$

$$2(300)(100) + 2(400)(125) = 2(500)\frac{dx}{dt}$$

$$2(3)(100) + 2(4)(125) = 10\frac{dx}{dt}$$

$$600 + 1000 = 10\frac{dx}{dt}$$

$$\boxed{\frac{dx}{dt} = 160 \text{ m/min}}$$

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t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\frac{-120-100}{8-2} = \boxed{\frac{-220}{6}} \text{ m/min}^2$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

Yes, The velocity drops from 40 m/min to -120 m/min

So at some point the velocity must have been at

-100 m/min

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Continue problem 4 on page 17.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

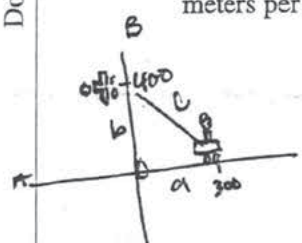
$$300 + \int_2^{12} v_A(t)\,dt \approx 300 + \frac{12-2}{6}\left(v_A(2) + 2v_A(5) + 2v_A(8) + v_A(12)\right)$$

$$= 300 + \frac{10}{6}(100 + 80 + -240 + -150)$$

$$= 300 + \frac{10}{6}(-210)$$

$$\boxed{= 300 - \frac{2100}{6}}$$ meters west of the origin Station

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.



$$a^2 + b^2 = c^2$$

$$300^2 + 400^2 = c^2$$

~~9600~~

$$900 + 1600 = c^2$$

$$2500 = c^2$$

$$500 = c$$

$\boxed{\text{500 meters per minute}}$

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Question 4

Overview

In this problem students were given a table of values of a differentiable function $v_A(t)$, the velocity of Train A, in meters per minute, for selected values of t in the interval $0 \le t \le 12$, where t is measured in minutes. In part (a) students were expected to know that the average acceleration of Train A over the interval $2 \le t \le 8$ is the average rate of change of $v_A(t)$ over that interval. The unit of the average acceleration is meters per minute per minute. In part (b) students were expected to state clearly that v_A is continuous because it is differentiable, and thus the Intermediate Value Theorem implies the existence of a time t between $t = 5$ and $t = 8$ at which $v_A(t) = -100$. In part (c) students were expected to show that the change in position over a time interval is given by the definite integral of the velocity over that time interval. If $s_A(t)$ is the position of Train A, in meters, at time t minutes, then $s_A(12) - s_A(2) = \int_2^{12} v_A(t)\,dt$, which implies that $s_A(12) = 300 + \int_2^{12} v_A(t)\,dt$ is the position at $t = 12$. Students approximated $\int_2^{12} v_A(t)\,dt$ using a trapezoidal approximation. In part (d) students had to determine the relationship between train A's position, train B's position, and the distance between the two trains. Students needed to put together several pieces of information from different parts of the problem and use implicit differentiation to determine the rate at which the distance between the two trains is changing at time $t = 2$.

Sample: 4A
Score: 9

The student earned all 9 points.

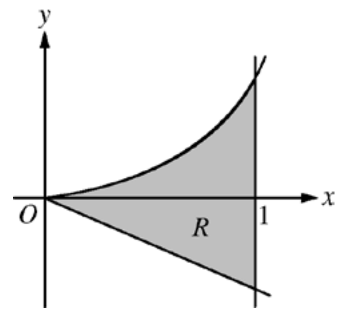
Sample: 4B
Score: 6

The student earned 6 points: no points in part (a), 2 points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student makes an arithmetic mistake in computing the average acceleration. In part (b) the student's work is correct. In part (c) the student earned the point for the position expression, but the trapezoidal sum is incorrect. The student is not eligible for the answer point. In part (d) the student's work is correct.

Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student encloses -100 within the required interval, but the student does not provide a reason. In part (c) the position expression is correct, but the trapezoidal sum is incorrect. The student is not eligible for the answer point. In part (d) the student's work did not earn any points.

Question 5



Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

(a) Area $= \int_0^1 \left(xe^{x^2} - (-2x) \right) dx$

$$= \left[\frac{1}{2}e^{x^2} + x^2 \right]_{x=0}^{x=1}$$

$$= \left(\frac{1}{2}e + 1 \right) - \frac{1}{2} = \frac{e+1}{2}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

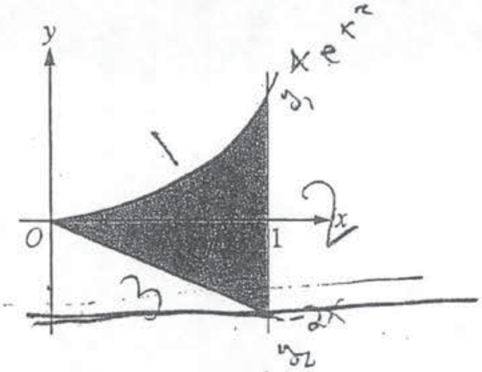
(b) Volume $= \pi \int_0^1 \left[\left(xe^{x^2} + 2 \right)^2 - (-2x + 2)^2 \right] dx$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

(c) $y' = \frac{d}{dx}\left(xe^{x^2} \right) = e^{x^2} + 2x^2 e^{x^2} = e^{x^2}\left(1 + 2x^2\right)$

Perimeter $= \sqrt{5} + 2 + e + \int_0^1 \sqrt{1 + \left[e^{x^2}\left(1 + 2x^2\right) \right]^2}\, dx$

$3 : \begin{cases} 1 : y' = e^{x^2}\left(1 + 2x^2\right) \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$



5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

$e^{x^2} + 2x^2e^{x^2}$

$$R = \int_0^1 x \cdot e^{x^2} dx - \int_0^1 -2x\, dx$$

$u = x^2,\ du = 2x$

$$= \frac{1}{2}\int_0^1 e^u du + 2\int_0^1 x\, dx$$

$$= \frac{1}{2} \cdot e^{x^2}\Big|_0^1 + x^2\Big|_0^1$$

$$= \frac{1}{2} \cdot e - \frac{1}{2} + 1 - 0$$

$$= \frac{e}{2} + \frac{1}{2} = \boxed{\frac{e+1}{2}}$$

Continue problem 5 on page 19.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

washer around x, interms of x

$$\boxed{V = \pi \int_0^1 \left[(xe^{x^2} + 2)^2 - (-2x + 2)^2 \right] dx}$$

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

Side 1: $\int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \sqrt{1 + (e^{x^2} + 2x^2 e^{x^2})^2}\, dx$

side 2: $y = 1 \cdot e^1 = e$, $y_2 = -2 \cdot 1 = -2$

length $= e + 2$

side 3: $\sqrt{a^2 + b^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$

$$\boxed{P = S_1 + S_2 + S_3 = (e+2) + \sqrt{5} + \int_0^1 \sqrt{1 + (e^{x^2} + 2x^2 e^{x^2})^2}\, dx}$$

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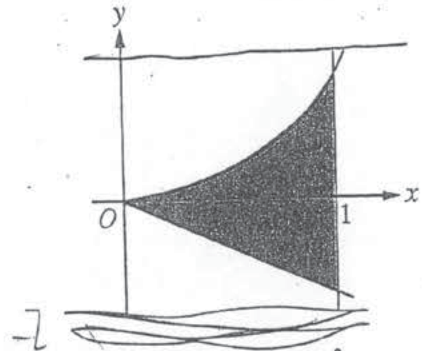

5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

$$\int_0^1 xe^{x^2}\,dx + \int_0^1 -2x\,dx = \text{area } R$$

$$\frac{1}{2}e^{x^2}\Big|_0^1 + -x^2\Big|_0^1$$

$$\left(\frac{1}{2}e^{1^2} - \frac{1}{2}e^{0^2}\right) + \left(-1^2 - (-0^2)\right) = \text{area}$$

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

$$\int_0^1 \left(\pi\left(xe^{x^2} - -2\right)^2 - \pi\left(-2x - -2\right)^2\right) dx$$

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

$$\int_0^1 \sqrt{1 + \left(x(2xe^{x^2}) + e^{x^2}\right)^2} + \int_0^1 \sqrt{1 + (-2)^2}\, dx$$

= perimeter

$y' = x\left(2xe^{x^2}\right) + e^{x^2}$

$y' = -2$

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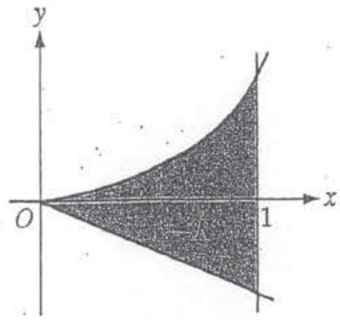

5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

$$R = \underbrace{\left(\int_0^1 xe^{x^2}\,dx\right)} + \left|\tfrac{1}{2}(-2\cdot 1)\right|$$

$u = x^2$

$du = 2x\,dx$

$\tfrac{1}{2}du = x\,dx$

$$\int_0^1 \tfrac{1}{2}e^u\,du = \tfrac{1}{2}e^u\Big|_0^1$$

$$= \tfrac{1}{2}e^{x^2}\Big|_0^1$$

$$R = \left[\tfrac{1}{2}e^{1^2} - \tfrac{1}{2}e^{0^2}\right] + |-1| = \left[\tfrac{1}{2}e - 1\right] + 1$$

$$R = \tfrac{1}{2}e - 1 + 1 = \tfrac{1}{2}e$$

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Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

$$V = \int_0^1 \left[(xe^{x^2})^2 - (-2x)^2 \right] dx$$

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

$$P = \left(\int_0^1 \sqrt{(2x^2 e^{x^2} + e^{x^2})^2 + 1}\, dx \right) + \sqrt{(1)^2 + (2)^2} + (2 + e)$$

$$y' = x \cdot e^{x^2} \cdot 2x + e^{x^2} \cdot 1$$
$$= 2x^2 e^{x^2} + e^{x^2}$$

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Question 5

Overview

In this problem students were given the graph of a region R bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$. In part (a) students were asked to find the area of R, requiring an appropriate integral setup and evaluation. Students needed to correctly evaluate $\int_0^1 \left(xe^{x^2} - (-2x)\right) dx$. Part (b) required students to find the volume of the solid generated when R is rotated about the horizontal line $y = -2$. Students needed to set up an integral where the integrand represents a cross-sectional area of a circular disc with inner radius $(-2x + 2)$ and outer radius $\left(xe^{x^2} + 2\right)$. This yielded the integral $\pi \int_0^1 \left(\left(xe^{x^2} + 2\right)^2 - (-2x + 2)^2\right) dx$. In part (c) students needed to write an expression involving one or more integrals that gives the perimeter of R. Students should have recognized that part of the perimeter involves finding the length of the curve $y = xe^{x^2}$ from $x = 0$ to $x = 1$ as well as the length of the two line segments. The resulting expression is $\int_0^1 \sqrt{1 + \left(e^{x^2}\left(1 + 2x^2\right)\right)^2}\, dx + \sqrt{5} + (2 + e)$.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 1 point in part (a), 3 points in part (b), and 2 points in part (c). In part (a) the student identifies the area of R as the sum of integrals from 0 to 1 of the given functions, rather than as the difference of these integrals. The student did not earn the first point. The student is eligible for and earned the second point for correctly antidifferentiating the stated integrands. The student is not eligible for the third point because both of the first 2 points were not earned. In part (b) the student's work is correct. In part (c) the student correctly differentiates xe^{x^2}, so the first point was earned. The student gives an integral equal to the length of the portion of the perimeter of R determined by xe^{x^2}, so the second point was earned. The student does not include the length of the vertical line segment in the expression for the perimeter of R, so the third point was not earned.

Question 5 (continued)

Sample: 5C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), and 1 point in part (c). In part (a) the student expresses the area of R as the sum of the integral from 0 to 1 of xe^{x^2} and a numeric value equal to the area of the triangular part of R below the x-axis. The student earned the first point. The student correctly antidifferentiates xe^{x^2}, so the second point was earned. The student makes an arithmetic error, so the third point was not earned. In part (b) the student does not present a correct integrand for the volume generated when R is rotated about the horizontal line $y = -2.$ The student is not eligible for any points in part (b). In part (c) the student correctly differentiates xe^{x^2}, so the first point was earned. The student gives an integral that is not equal to the length of the portion of the perimeter of R determined by xe^{x^2} because the derivative of xe^{x^2} is not squared. The student did not earn the second point. The student is not eligible for the third point because both of the first 2 points were not earned.

Question 6

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

(a) Let a_n be the nth term of the Taylor series.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}\,2^{n+1}(x-1)^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n+1}\,2^n(x-1)^n}$$
$$= \frac{-2n(x-1)}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{-2n(x-1)}{n+1}\right| = 2|x-1|$$

$$2|x-1| < 1 \Rightarrow |x-1| < \frac{1}{2}$$

The radius of convergence is $R = \frac{1}{2}$.

$3: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{determines radius of convergence} \end{cases}$

(b) The first three nonzero terms are
$2 - 4(x-1) + 8(x-1)^2$.

The general term is $(-1)^{n+1}2^n(x-1)^{n-1}$ for $n \ge 1$.

$3: \begin{cases} 2: \text{first three nonzero terms} \\ 1: \text{general term} \end{cases}$

(c) The common ratio is $-2(x-1)$.

$$f'(x) = \frac{2}{1-(-2(x-1))} = \frac{2}{2x-1} \text{ for } |x-1| < \frac{1}{2}$$

$$f(x) = \int \frac{2}{2x-1}\,dx = \ln|2x-1| + C$$

$f(1) = 0$
$\ln|1| + C = 0 \Rightarrow C = 0$
$f(x) = \ln|2x-1|$ for $|x-1| < \frac{1}{2}$

$3: \begin{cases} 1: f'(x) \\ 1: \text{antiderivative} \\ 1: f(x) \end{cases}$

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6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

This sum converges for x when $\lim_{n \to \infty} \left| \frac{\left(\frac{(-1)^{n+2}\, 2^{n+1}\, (x-1)^{n+1}}{n+1} \right)}{\left(\frac{(-1)^{n+1}\, 2^n\, (x-1)^n}{n} \right)} \right| < 1$

$\Rightarrow \lim_{n \to \infty} \left| \frac{(-1) \cdot 2 \cdot (x-1) \cdot n}{n+1} \right| < 1$

$\Rightarrow \left| 2(x-1) \right| < 1$

So $\left| (x-1) \right| < \frac{1}{2}$

Thus $\boxed{R = \frac{1}{2}}$

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Continue problem 6 on page 21.

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(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n (x-1)^n}{n} \quad \text{for } |x-1| < R$$

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n \cdot n (x-1)^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} 2^n \cdot (x-1)^{n-1}$$

The first three terms are: $2 - 4(x-1) + 8(x-1)^2$

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

$$f'(x) = 2\sum_{n=0}^{\infty} (-1)^n 2^n (x-1)^n = 2\sum_{n=0}^{\infty} (-2(x-1))^n$$

$$= 2 \cdot \frac{1}{1+2(x-1)} = \frac{2}{2x-1}$$

Then $f = \int f'$

$$= \int \frac{2}{2x-1}\,dx = \ln|2x-1| + C$$

$f(1) = 0$ because $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{n} (1-1)^n = 0$

$$\ln|2(1)-1| + C = 0$$

$$\Rightarrow \quad C = 0$$

$$\boxed{\text{So } f(x) = \ln|2x-1| \text{ for } |x-1| < R}$$

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6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

$$\lim_{n\to\infty}\left|\frac{\cancel{2^n}\,2\,(\cancel{x-1})^n(x-1)}{(n+1)}\cdot\frac{(n)}{(\cancel{x-1})^{\cancel{n}}(\cancel{2^n})}\right| =$$

$$\lim_{n\to\infty}\left|\frac{2n(x-1)}{(n+1)}\right|$$

$$|x-1|\lim_{n\to\infty} 2 < 1$$

$$|x-1| < \boxed{\frac{1}{2}}$$

$$-\frac{1}{2} < x-1 < \frac{1}{2}$$

$$\boxed{R = \frac{1}{2}}$$

Continue problem 6 on page 21

6B2

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(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

$$f = 2(x-1) - \frac{2^2}{2}(x-1)^2 + \frac{2^3}{3}(x-1)^3 \ldots (-1)^{n+1}\frac{2^n}{n}(x-1)^n$$

$$f' = 2 - \frac{2^2}{2}(2)(x-1) + \frac{2^3}{3}(3)(x-1)^2$$

$$\boxed{= 2 - 2^2(x-1) + 2^3(x-1)^2 \ldots (-1)^{n+1}2^n(x+1)^{n-1}}$$

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

$$\frac{a}{1-r} \qquad a = 2 \qquad r = \frac{1}{2(x-1)}$$

$$f' = \frac{2}{1-\frac{1}{2(x-1)}} = \frac{2}{\frac{2(x-1)-1}{2(x-1)}} \boxed{= \frac{4(x-1)}{2(x-1)-1} = f'}$$

$$f = \frac{2(x-1)}{1-\frac{1}{(x-1)}}$$

$$\cancel{\frac{2(x-1)}{\frac{2^2}{2}(x-1)^2}} \qquad \cancel{\frac{\frac{2^2}{2}(x-1)^2}{\frac{2^3}{3}(x-1)^3}}$$

$$f = \frac{2(x-1)}{\frac{(x-1)-1}{(x-1)}} = \boxed{\frac{2(x-1)^2}{(x-1)-1} = f}$$

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6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

$$\lim_{n \to \infty} \left| \frac{(-1)^{n+2} \frac{2^{n+1}}{n+1} (x-1)^{n+1}}{(-1)^{n+1} \frac{2^n}{n} (x-1)^n} \right|$$

$$\lim_{n \to \infty} \left| \frac{(-1)(2)(x-1)(n)}{(n+1)} \right|$$

$$\lim_{n \to \infty} \left| \frac{2n(x-1)}{n+1} \right| = 2$$

$$-2 < (x-1) < 2$$

$$-1 < (x-1) < 3 \qquad \frac{3+1}{2} = 2$$

$$\boxed{R = 2}$$

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Continue problem 6 on page 21.

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(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

$$\frac{2}{1}(x-1) - \frac{4}{2}(x-1)^2 + \frac{8}{3}(x-1)^3 \ldots + -1^{n+1}\frac{2^n}{n}(x-1)^n$$

$$\boxed{2 - 4(x-1) + 8(x-1)^2 + \ldots + (-1)^n \; 2n(x-1)^{n-1}}$$

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(2n+2)(x-1)^n}{(-1)^n(2n)(x-1)^{n-1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(-1)(2n+2)(x-1)}{2n} \right| = 1$$

$$-1 < x-1 < 1$$

$$0 < x < 2 \qquad \frac{2}{2} = 1$$

$$\boxed{R = 1}$$

GO ON TO THE NEXT PAGE.

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Question 6

Overview

In this problem students were given a Taylor series for a function f about $x = 1$. In part (a) students were asked to find the radius of convergence of this Taylor series. It was expected that students would use the ratio test to determine that the radius of convergence is $\frac{1}{2}$. In part (b) students needed to differentiate the series term-by-term to find the first three nonzero terms and the general term of the Taylor series for f'. In part (c) students were told that the Taylor series for f' is a geometric series. Students needed to know that finding the sum of that series requires dividing the first term of the series by the difference of 1 and the common ratio. This results in $f'(x) = \frac{2}{2x-1}$. Students were also asked to find f. This required integrating $f'(x)$ to find $f(x) = \ln|2x - 1| + C$. In order to evaluate the constant of integration, students needed to use the initial condition that $f(1) = 0$ which yields $f(x) = \ln|2x - 1|$ for $|x - 1| < \frac{1}{2}$.

Sample: 6A
Score: 9

The student earned all 9 points. In part (c) the student does not need to qualify the closed form expression for $f(x)$ with "for $|x - 1| < R$."

Sample: 6B
Score: 6

The student earned 6 points: 3 points in part (a), 3 points in part (b), and no points in part (c). In parts (a) and (b), the student's work is correct. In part (b) the student writes the first three nonzero terms and the general term of the Taylor series of the original function f and then differentiates to find the required first three terms and the general term of the Taylor series of the derivative of f. Because the question asks students to find the first three nonzero terms and the general term of the Taylor series for f', the student is not penalized for omitting plus signs and an ellipsis in the boxed answer in part (b). In part (c) the student misidentifies the constant ratio in the geometric series. Because the first point was not earned, the student is not eligible for the third point. The student does not antidifferentiate correctly, so the second point was not earned.

Sample: 6C
Score: 3

The student earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the student earned 1 point with an appropriate ratio. The student's announced limit and conclusion are incorrect. In part (b) the student earned 2 points for giving the correct first three nonzero terms. The student's general term is incorrect. In part (c) the student does not provide an expression for f', so the student is not eligible for any points.